Int. J. Solids Structures Vol. 26, No. 8, pp. 887–900, 1990
Printed in Great Britain.

0020–7683/90 \$3.00 + .00

A VARIATIONAL PRINCIPLE FOR FINITE PLANAR DEFORMATION OF STRAIGHT SLENDER ELASTIC BEAMS

M. SAJE
Edvard Kardelj University in Ljubljana, Faculty for Architecture, Civil Engineering and Surveying, Jamova 2, 61000 Ljubljana, Yugoslavia

(*Received* 3 *June* 1989; *in revised form* 30 *August* 1989)

Abstract—The objective of this paper is to present a finite element formulation for finite deformation analysis of a slender straight elastic in-plane beam. The formulation is based on a modified Hu–Washizu variational principle in which exact nonlinear kinematic equations are taken into account. Only the rotation in the interior of an element needs to be approximated. Thus the ambiguities concerning the order of polynomial approximations for different field variables, e.g. longitudinal and lateral displacements, are avoided. The Euler–Lagrange equations of this principle are, among others, exact kinematic and equilibrium conditions for the beam. The solution capabilities are illustrated with numerical examples. Two different finite elements are examined: with 3rd-order interpolation polynomials for the rotation, and with 5th-order polynomials. Excellent convergence and accuracy of both elements is demonstrated. The results indicate that the accuracy of the present elements is not notably influenced by the length of the element and the order of numerical integration (if it is greater than a minimal value). Only one load step was sufficient in calculations to achieve the results.

1. INTRODUCTION

Most of the finite element large deformation analyses of beams consider the beam as a one-dimensional continuum and use the conventional displacement approach, applying a virtual work principle or a principle of minimum potential energy. In such analyses, the displacement components are approximated by polynomials, employing the displacements and their first derivatives as the nodal degrees of freedom of the element (Epstein and Murray, 1976). However, the derivatives of the displacements are not convenient nodal parameters for frame analysis and must be replaced by nodal rotations in production computer programs. Since this is not likely to be possible for exact nonlinear kinematic relations, "small" terms in kinematic equations are often neglected, so that the derivative of lateral displacement with respect to the longitudinal coordinate becomes a linear function of the rotation. In case of very large deformations such simplification, however, requires the introduction of a rotated local coordinate system, a relatively small length of the finite element and other improvements. Moreover, the displacement approach is inferior to the mixed approach, especially for nonlinear materials, as discussed by Banovec (1981), Noor and Peters (1981) and Karamanlidis (1988).

The objective of this paper is to present a finite element formulation for finite planar deformations and rotations of a slender straight elastic beam, based on a modified Hu–Washizu variational principle (Washizu, 1981) in which *exact* nonlinear kinematic equations are taken into account. Only the rotation in the interior of an element needs to be approximated. The ambiguities concerning the order of polynomial approximations for different field variables, e.g. longitudinal and lateral displacements, are thus avoided. The Euler–Lagrange equations of this principle are, among others, exact kinematic and equilibrium conditions for the finite deformations of the beam. The solution capabilities are illustrated with numerical examples.

2. KINEMATIC AND CONSTITUTIVE RELATIONS

Let the locus of the centroids of the cross-section of the undeformed beam element be a straight line, and let it coincide with the z^1 axis of the Cartesian coordinate system (z^1, z^2, z^3) with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ as the unit base vectors. Let the cross-section A of the beam in the

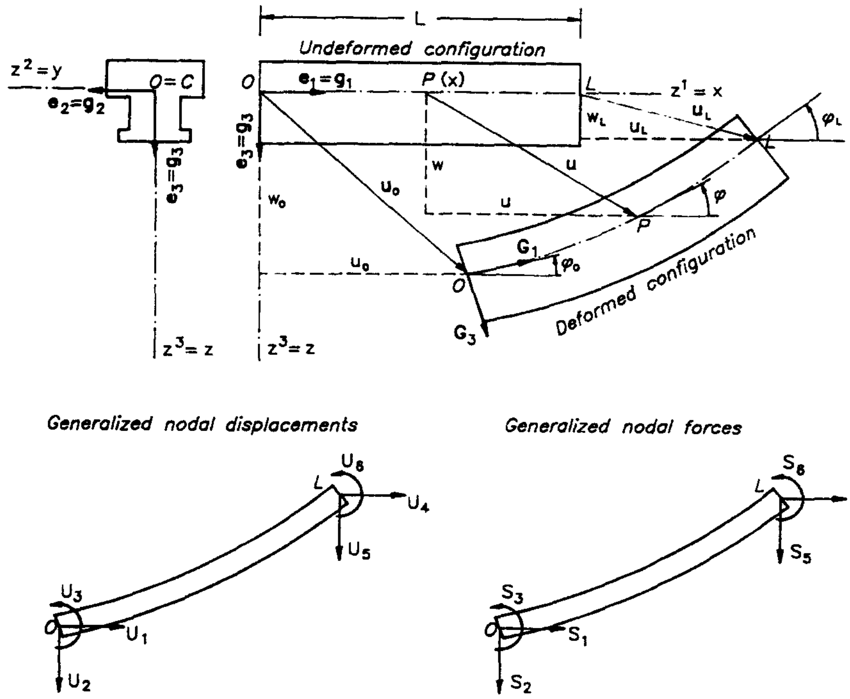


Fig. 1. The deformation of a beam element.

coordinate plane z^1 = const be symmetric with respect to the coordinate plane $z^2 = 0$ (Fig. 1). The Lagrangian description is used with the initial undeformed configuration taken as reference. A material particle is identified by material coordinates $x^1 \equiv x, x^2 \equiv y$ and $x^3 \equiv z$, which coincide with the Cartesian coordinates z^1, z^2, z^3 in the reference configuration. The material base vectors in the reference and in the current deformed configuration are denoted by $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ and $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$.

The components u and w of the displacement vector $\mathbf{u}(x)$ along the centroid axis

$$\mathbf{u} = u\mathbf{g}_1 + w\mathbf{g}_3 \tag{1}$$

are related to the deformation functions $\varepsilon(x)$ and $\varphi(x)$, where ε is extensional strain of the centroid axis and φ its rotation, by the kinematic equations

$$1 + u' - (1+\varepsilon)\cos\varphi = 0 \tag{2}$$

$$w' + (1+\varepsilon)\sin\varphi = 0, \tag{3}$$

as shown e.g. in Reissner (1972) or Saje and Srpčič (1985). In eqns (2)–(3) prime (′) denotes the derivative with respect to the longitudinal coordinate x.

The constitutive law for material is assumed to be given by a linear elastic relation between the longitudinal true stress component, σ, and the associated extensional strain, D

$$D = \varepsilon + z\varphi'$$

of the generic particle (x, z) of a beam:

$$\sigma = ED. \tag{4}$$

E is the elastic modulus of material. Integration of stresses over the cross-section gives the axial force, N, and bending moment, M, of the cross-section

$$N = \int_A \sigma \, dA = EA\varepsilon \tag{5}$$

$$M = \int_A z\sigma \, dA = EI\varphi'. \tag{6}$$

Here A is the area and I the moment of inertia of the cross-section with respect to the y axis.

3. GENERALIZED TOTAL POTENTIAL ENERGY FUNCTIONAL

Consider a straight elastic beam of initial length L, subjected to prescribed external distributed loads and moment per unit of the undeformed length of the centroid axis, p_x, p_z and m_y, and to point loads at both boundaries at $x = 0$ and $x = L$. The following generalized total potential energy functional, E_p, for the planar deformation of the beam is proposed:

$$E_p(u, w, \varepsilon, \varphi, R_1, R_2, U_k) = \tfrac{1}{2}\int_0^L AE\varepsilon^2 \, dx + \tfrac{1}{2}\int_0^L EI\varphi'^2 \, dx - \int_0^L p_x u \, dx - \int_0^L p_z w \, dx - \int_0^L m_y \varphi \, dx - E_p^B(U_k) + \int_0^L R_1[1+u'-(1+\varepsilon)\cos\varphi] \, dx + \int_0^L R_2[w'+(1+\varepsilon)\sin\varphi] \, dx, \tag{7}$$

where the kinematic constraints (2)–(3) have been taken into account by the Lagrangian multipliers $R_1(x)$ and $R_2(x)$. In eqn (7) U_k $(k = 1, 2, 3, \ldots, 6)$ denotes generalized displacements at the two ends of the beam, and E_p^B is the potential of the boundary forces, S_k (Fig. 1). E_p is a function of six independent functions of x, i.e. u, w, ε, φ, R_1, R_2, and six independent generalized nodal displacements, U_k.

By equating the variation of the functional (7) with respect to its arguments to zero

$$\delta E_p = \frac{\partial E_p}{\partial \varepsilon}\delta\varepsilon + \frac{\partial E_p}{\partial \varphi}\delta\varphi + \frac{\partial E_p}{\partial u}\delta u + \frac{\partial E_p}{\partial w}\delta w + \frac{\partial E_p}{\partial R_1}\delta R_1 + \frac{\partial E_p}{\partial R_2}\delta R_2 + \frac{\partial E_p}{\partial U_k}\delta U_k = 0$$

and integrating by parts the terms with u', w' and φ', the Euler–Lagrange equations of the functional are derived: the kinematic constraints (2) and (3), the differential equations

$$AE\varepsilon - R_1\cos\varphi + R_2\sin\varphi = 0 \tag{8}$$

$$EI\varphi'' - (1+\varepsilon)(R_1\sin\varphi + R_2\cos\varphi) + m_y = 0 \tag{9}$$

$$R_1' + p_x = 0 \tag{10}$$

$$R_2' + p_z = 0 \tag{11}$$

and the boundary conditions at $x = 0$ and $x = L$:

$$x = 0: \quad S_1 + R_1^0 = 0 \tag{12}$$

$$S_2 + R_2^0 = 0 \tag{13}$$

$$S_3 + EI\varphi_0' = 0 \tag{14}$$

$$x = L: \qquad S_4 - R_1^L = 0 \tag{15}$$

$$S_5 - R_2^L = 0 \tag{16}$$

$$S_6 + EI\varphi'_L = 0. \tag{17}$$

By integrating differential equations (10) and (11) we obtain

$$R_1(x) = R_1^0 - \int_0^x p_x(\xi)\,\mathrm{d}\xi = R_1(R_1^0, x) \tag{18}$$

$$R_2(x) = R_2^0 - \int_0^x p_z(\xi)\,\mathrm{d}\xi = R_2(R_2^0, x). \tag{19}$$

We have assumed that the tractions p_x and p_z are integrable functions of x. Functions $R_1(x)$ and $R_2(x)$ are thus expressed in terms of their values R_1^0 and R_2^0 at $x = 0$. We insert eqns (18)–(19) into the functional (7), eliminate ε from the functional by employing eqn (8)

$$\varepsilon = \frac{1}{AE}(R_1 \cos\varphi - R_2 \sin\varphi) = \varepsilon(R_1^0, R_2^0, \varphi, x) \tag{20}$$

and integrate by parts terms with u' and w' and thus obtain a new functional, E_p^*:

$$E_p^*(\varphi, R_1^0, R_2^0, U_k) = \tfrac{1}{2}\int_0^L AE\varepsilon^2\,\mathrm{d}x + \tfrac{1}{2}\int_0^L EI\varphi'^2\,\mathrm{d}x - \int_0^L m_y\varphi\,\mathrm{d}x$$

$$+ \int_0^L R_1[1-(1+\varepsilon)\cos\varphi]\,\mathrm{d}x + \int_0^L R_2(1+\varepsilon)\sin\varphi\,\mathrm{d}x - (S_1 + R_1^0)U_1$$

$$-(S_2 + R_2^0)U_2 - S_3U_3 - (S_4 - R_1^L)U_4 - (S_5 - R_2^L)U_5 - S_6U_6. \tag{21}$$

This functional is a function of *only one function*, i.e. the rotation $\varphi(x)$, and of *eight discrete values*, i.e. boundary displacements $U_1 = u_0$, $U_2 = w_0$, $U_4 = u_L$ and $U_5 = w_L$, boundary rotations $U_3 = \varphi_0$ and $U_6 = \varphi_L$, and Lagrangian multipliers at $x = 0$, i.e. R_1^0 and R_2^0. In such a functional only the variation of rotations must be assumed, while no approximations need be made regarding the variations of u, w and ε in the interior of the element. This is a considerable advantage in the finite element implementation of the variational principle. Because the first derivative of the rotation is the highest derivative in the functional, only the C^0 continuity must be given to the approximating function.

It has to be emphasised that no restrictions concerning the size of rotations, extensional strain, or displacements have been set. A conceptually similar functional was proposed by Banovec (1981) for large displacements and moderate rotations analysis of elastic–plastic beams, where, however, the kinematic constraints (2)–(3) were simplified.

4. FINITE ELEMENT FORMULATION

Consider a beam element of initial length L with a nodal system consisting of M equidistant nodes (Fig. 2). For the approximation of the distribution of the rotation along the beam, polynomial interpolation functions, $I_k(x)$, of degree $M-1$ are employed. The variation of the rotation $\varphi(x)$ over the element is expressed by the interpolation equation

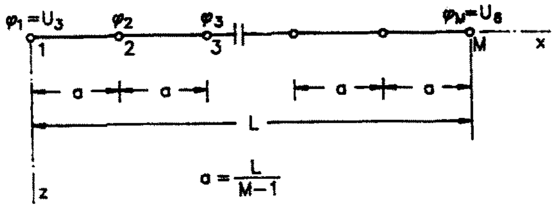


Fig. 2. The beam finite element.

$$\varphi(x) = \sum_{k=1}^{M} I_k(x)\varphi_k, \tag{22}$$

where φ_k $(k = 1, 2, \ldots, M)$ are nodal rotations of the element, φ_1 being equal to U_3 and φ_M to U_6 (see Fig. 2). Inserting expression (22) into the functional (21), varying the functional with respect to the boundary displacements and nodal rotations, R_1^0 and R_2^0, and equating the variation of the functional to zero, the following system of algebraic equations for the element is obtained:

$$L_1(\varphi_m, R_j^0, U_1, U_4) = -\int_0^L (1+\varepsilon)\cos\varphi\,\mathrm{d}x + U_4 - U_1 + L = 0 \tag{23}$$

$$L_2(\varphi_m, R_j^0, U_2, U_5) = -\int_0^L (1+\varepsilon)\sin\varphi\,\mathrm{d}x + U_5 - U_2 = 0 \tag{24}$$

$$S_k^*(\varphi_m, R_j^0) = \int_0^L \{EI\varphi' I_k' + [(1+\varepsilon)(R_1 \sin\varphi + R_2\cos\varphi) - m_y]I_k\}\,\mathrm{d}x$$

$$k = 1, 2, 3, \ldots, M; \quad m = 1, 2, 3, \ldots, M; \quad j = 1, 2. \tag{25}$$

$$x = 0: \quad S_1 + R_1^0 = 0 \tag{26}$$

$$S_2 + R_2^0 = 0 \tag{27}$$

$$x = L: \quad S_4 - R_1^L = 0 \tag{28}$$

$$S_5 - R_2^L = 0, \tag{29}$$

where

$$S_3 = S_1^* \tag{30a}$$

$$S_6 = S_6^* \tag{30b}$$

$$S_k^* = 0 \quad \text{for } k = 2, 3, 4, 5. \tag{31}$$

Equations (23)–(29) constitute a system of $M+6$ nonlinear algebraic equations for $M+6$ unknowns $\varphi_1, \varphi_2, \ldots, \varphi_M, U_1, U_2, U_4, U_5, R_1^0$ and R_2^0. The unknowns $\varphi_2, \ldots, \varphi_{M-1}, R_1^0$ and R_2^0 are *internal* degrees of freedom of the element, while $U_1, U_2, U_3 = \varphi_1, U_4, U_5$ and $U_6 = \varphi_M$ are its *external* degrees of freedom. The system is solved iteratively, employing the Newton method. In an iteration step, Newton's increments of the internal degrees of freedom $(\Delta\varphi_2, \ldots, \Delta\varphi_{M-1}, \Delta R_1^0, \Delta R_2^0)$ are first expressed in terms of increments of the external degrees $(\Delta U_1, \Delta U_2, \Delta U_3 = \Delta\varphi_1, \Delta U_4, \Delta U_5, \Delta U_6 = \Delta\varphi_M)$ from a linearized system of eqns (23)–(25):

$$\sum_{m=1}^{M} \frac{\partial L_1}{\partial \varphi_m} \Delta\varphi_m + \sum_{j=1}^{2} \frac{\partial L_1}{\partial R_j^0} \Delta R_j^0 + \frac{\partial L_1}{\partial U_1} \Delta U_1 + \frac{\partial L_1}{\partial U_4} \Delta U_4 = -L_1(\varphi_m, R_j^0, U_1, U_4) \quad (32)$$

$$\sum_{m=1}^{M} \frac{\partial L_2}{\partial \varphi_m} \Delta\varphi_m + \sum_{j=1}^{2} \frac{\partial L_2}{\partial R_j^0} \Delta R_j^0 + \frac{\partial L_2}{\partial U_2} \Delta U_2 + \frac{\partial L_2}{\partial U_5} \Delta U_5 = -L_2(\varphi_m, R_j^0, U_2, U_5) \quad (33)$$

$$\sum_{m=1}^{M} \frac{\partial S_i^*}{\partial \varphi_m} \Delta\varphi_m + \sum_{j=1}^{2} \frac{\partial S_i^*}{\partial R_j^0} \Delta R_j^0 = -S_i^*(\varphi_m, R_j^0) \quad i = 1, 2, \ldots, M. \quad (34)$$

The coefficients of the system of equations (32)–(34) are shown in the Appendix. Gaussian integration is employed for the determination of the integrals in (32)–(34).

Eliminating increments of internal nodal rotations from eqns (34) gives $\Delta\varphi_m$ ($m = 2, 3, \ldots, M-1$) as a linear combination of increments of ΔR_1^0, ΔR_2^0 and generalized nodal displacements ΔU_k:

$$\Delta\varphi_m = \sum_{j=1}^{2} F_{mj} \Delta R_j^0 + \sum_{k=1}^{6} G_{mk} \Delta U_k + f_m \quad (m = 2, 3, \ldots, M-1). \quad (35)$$

Inserting eqn (35) in (32)–(33), the increments ΔR_1^0 and ΔR_2^0 are obtained in terms of ΔU_k:

$$\Delta R_j^0 = \sum_{k=1}^{6} H_{jk} \Delta U_k + r_j \quad (j = 1, 2). \quad (36)$$

Reorganization and linearization of eqns (26)–(30) gives

$$\Delta S_1 = -\Delta R_1^0 \quad (37)$$

$$\Delta S_2 = -\Delta R_2^0 \quad (38)$$

$$\Delta S_3 = \Delta S_1^* \quad (39)$$

$$\Delta S_4 = \Delta R_1^L \quad (40)$$

$$\Delta S_5 = \Delta R_2^L \quad (41)$$

$$\Delta S_6 = \Delta S_6^*. \quad (42)$$

According to eqns (18) and (19),

$$\Delta R_1^L = \Delta R_1^0$$

and

$$\Delta R_2^L = \Delta R_2^0.$$

Inserting eqns (35)–(36) and (25) into eqns (37)–(42) yields increments in boundary forces ΔS_m expressed by increments of generalized boundary displacements, ΔU_k:

$$\Delta S_m = \sum_{k=1}^{6} K_{mk} \Delta U_k + l_m \quad (m = 1, 2, \ldots, 6). \quad (43)$$

Here K_{mk} are the components of a tangent stiffness matrix and l_m the components of a tangent load vector of the beam element.

Equation (43) presents the relation between the increments of generalized boundary forces, ΔS_m, and their energy complements, increments of generalized boundary displacements, ΔU_k, for an element in a coordinate system (z^1, z^2, z^3) of the element. The corresponding relation in another coordinate system is obtained by the coordinate transformation.

The increments of boundary displacements are determined iteratively from nodal equilibrium equations of a structure. Let the structure have "E" elements and "N" nodes, the latter being subjected to generalized external forces, $F_k^{(n)}$ ($n = 1, 2, \ldots, N$; $k = 1, 2, 3$). Equilibrium equations for an isolated node "n" in which E_n elements are connected, are:

$$F_k^{(n)} - \sum_{e=1}^{E_n} S_k^{(n)} = 0 \quad (n = 1, \ldots, N;\ k = 1, 2, 3). \tag{44}$$

A linearization of (44), in Newton's sense, gives

$$\sum_{e=1}^{E_n} \Delta S_k^{(n)} = -F_k^{(n)} + \sum_{e=1}^{E_n} S_k^{(n)} \quad (n = 1, \ldots, N;\ k = 1, 2, 3). \tag{45}$$

Introduction of (43) and (25)–(29) for each connecting element into (45) yields a system of linear equations from which, upon imposing displacement boundary conditions, the increments of generalized nodal displacements, $\Delta U_i^{(n)}$, for the whole structure are obtained. ΔR_j^0 and increments of the internal nodal rotations $\Delta\varphi_m$ for elements are then determined *a posteriori* from eqns (36) and (35). New approximations for $U_i^{(n)}$, $\varphi_m^{(e)}$ and $R_j^{0(e)}$ are then

$$\text{NEW}U_i^{(n)} = U_i^{(n)} + \Delta U_i^{(n)} \quad (i = 1, 2, 3;\ n = 1, \ldots, N) \tag{46}$$

$$\text{NEW}\varphi_m^{(e)} = \varphi_m^{(e)} + \Delta\varphi_m^{(e)} \quad (m = 2, 3, \ldots, M-1;\ e = 1, 2, \ldots, E) \tag{47}$$

$$\text{NEW}R_j^{0(e)} = R_j^{0(e)} + \Delta R_j^{0(e)} \quad (j = 1, 2;\ e = 1, 2, \ldots, E). \tag{48}$$

The iteration cycle is repeated until the required accuracy for unknowns is achieved. Zero values are assumed as starting approximations for the quantities $U_i^{(n)}$, $\varphi_m^{(e)}$ and $R_j^{0(e)}$.

After the iteration has been completed and the unknowns determined, the internal forces at the element boundaries ($x = 0$ and $x = L$) are determined using eqns (26)–(29) for axial and shear forces and eqn (25) for the bending moment. The distribution of internal forces in the interior of a beam element is obtained by application of the equilibrium equations rather than the constitutive equations. The equilibrium equations for axial and shear forces, N and Q, at the cross-section $x = \text{const}$, for example, give (Saje and Srpčič, 1985)

$$N(x) = R_1 \cos\varphi - R_2 \sin\varphi \tag{49}$$

$$Q(x) = R_1 \sin\varphi + R_2 \cos\varphi. \tag{50}$$

A similar equation is derived from the equilibrium equations for the bending moment $M(x)$.

The distribution of the displacement components, u and w, over the element and a corresponding deformed shape of the beam are found by integration of eqns (2)–(3). We have applied Gaussian integration of order equal to that employed for the determination of the element tangent stiffness matrix and load vector. This assures that the accuracy in deformed shape determination is equal to that achieved in Newton's iterative solution of eqns (23)–(24).

The lowest reasonable order of Gaussian integration can be estimated by inspecting the integrands of eqns (A1)–(A13) of the Appendix and eqns (23)–(25) for a simplified situation, where we can assume sufficiently small rotations ($|\varphi| \approx 0$) and constant axial and shear forces along the beam. Under such conditions the critical integrand appears to be the one in eqn (A13) containing a product of interpolation functions $I_l I_m$, which is a polynomial of degree $2(M-1)$. To make the integration of such a polynomial accurate the Gaussian integration of order M is required. The influence of the integration order on the accuracy of the solution will be discussed further on.

5. EXAMPLE PROBLEMS

We consider a cantilever subjected to various types of loadings and investigate the influence of the degree of interpolation polynomials and the order of Gaussian integration on the accuracy and the convergence of the numerical solution. Two different elements have been examined: element F3 with 3rd-order interpolation polynomials for the rotation, and element F5 with 5th-order polynomials. The lowest recommended orders of Gaussian integration for the two elements are 4 and 6, respectively. In our calculations the orders of Gaussian integration between 3 and 10 have been chosen.

The cantilever has been modeled by only *one element. One loading step from the initial undeformed configuration to the current deformed configuration is applied* in all loading cases. The iteration is stopped when the Euclidean norm of Newton's increments in relative end displacements (U/L, W/L) and rotations (ϕ) is smaller than 10^{-6}. A double precision arithmetic (approximately 16 digits accuracy) was used in the computer calculations.

The numerical solutions are compared to the analytical ones for incompressible elastica (Saje and Srpčič, 1985; Mattiasson, 1981; Pflüger, 1964). To assure incompressibility of the centroid axis of the beam a large value for the axial stiffness of the beam has been chosen.

Example 1: *A cantilever subjected at its free end to a moment* M

Consider a cantilever of length L subjected to a moment M at its free end. The descriptive data for the problem are given in Table 1. The exact solution (e.g. Saje and Srpčič, 1985) shows that the deformed shape of the cantilever is part of a circle with curvature κ,

$$\kappa = \varphi' = \frac{M}{EI}. \tag{51}$$

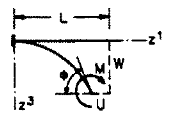
Table 1 compares the computed values of the relative displacements (U/L, W/L) and the rotation (ϕ) at the free end of the cantilever with the exact values. If a sufficiently high order of Gaussian integration is used, accuracy up to six decimals is obtained for both elements, F3 and F5, and for the reduced end moment, $\bar{M}$, up to the very high value 4π (the beam winds around the support twice). Whatever the magnitude of the end moment only three iterations are needed to achieve this accuracy.

Example 2: *A cantilever subjected at its free end to a force* F

The errors in the displacement components and rotation at the free end of the cantilever, obtained with elements F3 and F5, are presented in Table 2. Comparisons are made with the analytic solutions of Mattiasson (1981). With reduced forces, $\bar{F}$, ranging from a small value 0.2 (end deflection is about 7% of beam length) to a very high value 10 (end deflection is about 81% of the length) and with the orders of Gaussian integration from 3 to 10, the highest relative end displacement or rotation error with respect to the exact solution is 6‰ for the F3 element and 0.06‰ for the F5 element. Three to five iterations were needed.

For reduced forces not bigger than $\bar{F} = 1$, the solutions given by both elements are identical to the analytic solutions of Mattiasson (1981). For this range of $\bar{F}$, the absolute magnitudes of the rotations are smaller than 0.5 (approximately 30°), which, according to the classification of Stein (1982), defines *the bound between large and finite rotations*. We may therefore conclude that both elements (F3 and F5) give exact values if rotations are large.

Table 1. A cantilever subjected at its free end to a moment M. One element, one load step



$AE = 10^{21}$
$IE = 10$
$L = 1$
$\bar{M} = ML/EI = 0.1M$

$\bar{M}$	e	n_G	n_I	ε_u (‰)	ε_w (‰)	ε_ϕ (‰)
$\pi/4$	F3	3	3	**0.00**	**0.00**	**0.00**
		10	3	**0.00**	**0.00**	**0.00**
	F5	5	3	**0.00**	**0.00**	**0.00**
		10	3	**0.00**	**0.00**	**0.00**
	EXACT			*−0.099684*	*0.372923*	*0.785398*
π	F3	3	3	**0.00**	**0.69**	**0.00**
		10	3	**0.00**	**0.00**	**0.00**
	F5	5	3	**0.00**	**0.00**	**0.00**
		10	3	**0.00**	**0.00**	**0.00**
	EXACT			*−1.000000*	*0.636620*	*3.141593*
4π	F3	3	3	**−530.00**	**0.00**	**0.00**
		4	3	**125.73**	**0.00**	**0.00**
		10	3	**0.00**	**0.00**	**0.00**
	F5	5	3	**−16.67**	**0.00**	**0.00**
		6	3	**1.42**	**0.00**	**0.00**
		10	3	**0.00**	**0.00**	**0.00**
	EXACT			*−1.000000*	*0.000000*	*12.566371*
*EXACT SOLUTION**				U/L	W/L	ϕ

e type of element, F3 or F5.
n_G order of Gaussian integration.
n_I number of Newton's iterations.
ε_Q relative error in quantity Q in comparison to exact solution, Q_E, $\varepsilon_Q = 1000(Q-Q_E)/Q_E$, in ‰.
* Saje and Srpčič (1985).

Table 3 presents the values of the Euclidean norms of solution increments and of the residual vectors throughout the iteration for $\bar{F} = 10$. A good (roughly quadratic) convergence rate may be observed. Also some other numerical experiments with various loading conditions show a good convergence rate for elements F3 and F5.

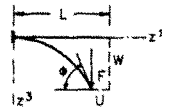
The order of Gaussian integration may have some influence on the accuracy of the solution, especially with element F3, but it does not influence the number of iterations. If the order of Gaussian integration is equal to 10, the differences between the calculated end displacements and rotations and their exact values for $\bar{F} = 10$ are less than 1.25‰ for element F3 (accuracy to three significant figures) and less than 0.02‰ for element F5 (accuracy to five significant figures).

Example 3: *A buckling problem*

We consider the buckling of a geometrically perfect elastic beam subjected to a compressive force P at its end. The critical values, P_B, of the compressive force P for the first buckling mode of the beam have been calculated using the elements F3 and F5. The critical load has been found in an iterative way by seeking, iteratively, a force which makes the constrained element and structural tangent stiffness matrices singular. During the deformation up to the critical load, the rotation of the centroid axis remains zero and the axial force is constant throughout the beam, which substantially simplifies the integrals in the element tangent stiffness matrix and load vector. Examining these integrals shows that Gaussian integration gives *exact* results if the integration order in elements F3 and F5 is equal to 4 and 6, respectively.

Table 4 shows the error in the buckling loads for three differently supported beams: (1) a cantilever beam; (2) a clamped–simply supported beam; and (3) a beam clamped at

Table 2. A cantilever subjected at its free end to a force F. One element, one load step



$AE = 10^{21}$
$IE = 10$
$L = 1$
$\bar{F} = FL^2/EI = 0.1F$

$\bar{F}$	e	n_G	n_I	ε_u (‰)	ε_w (‰)	ε_ϕ (‰)
0.2	**F3**	3	3	**0.00**	**0.00**	**0.00**
		10	3	**0.00**	**0.00**	**0.00**
	F5	5	3	**0.00**	**0.00**	**0.00**
		10	3	**0.00**	**0.00**	**0.00**
	EXACT			*−0.00265*	*0.06636*	*0.09964*
5.0	**F3**	3	5	**−2.55**	**−2.33**	**−0.14**
		4	5	**0.21**	**−0.32**	**0.15**
		10	5	**0.18**	**−0.24**	**0.18**
	F5	5	5	**0.00**	**0.00**	**0.00**
		10	5	**0.00**	**0.00**	**0.00**
	EXACT			*−0.38763*	*0.71379*	*1.21537*
10.0	**F3**	3	5	**−6.03**	**−4.33**	**1.91**
		4	5	**−0.07**	**−1.70**	**1.06**
		10	5	**0.17**	**−1.25**	**1.21**
	F5	5	5	**0.02**	**−0.04**	**−0.01**
		6	5	**−0.02**	**0.00**	**−0.01**
		10	5	**−0.02**	**−0.01**	**−0.01**
	EXACT			*−0.55500*	*0.81061*	*1.43029*
*EXACT SOLUTION**				*U/L*	*W/L*	ϕ

e type of element, F3 or F5.
n_G order of Gaussian integration.
n_I number of Newton's iterations.
ε_Q relative error in quantity Q in comparison to exact solution, Q_E, $\varepsilon_Q = 1000(Q-Q_E)/Q_E$, in ‰.
* Mattiasson (1981).

Table 3. Euclidean norms of iteration steps for a beam subjected at its free end to a force $\bar{F} = 10$

Iteration number i	Euclidean norm* of Solution increment	Euclidean norm* of Residual vector
1	6.0×10^{1}	1.1×10^{5}
2	2.5×10^{1}	1.9×10^{5}
3	1.1×10^{-1}	6.8×10^{2}
4	1.3×10^{-4}	7.8×10^{-1}
5	1.1×10^{-9}	6.4×10^{-6}

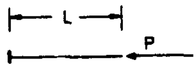
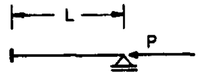
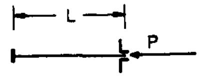
Element F5.
Order of Gaussian integration: $n_G = 10$.
* |v| = SQRT (vv).

both ends. The relative error in the buckling load of element F3 ranges from 0.14‰ in a cantilever to 63.87‰ in a doubly clamped beam. Element F5 is much more accurate, exhibiting a relative error of only 0.59‰ in the most severe case (3).

Example 4: A cantilever subjected at its free end to a force F. *Assessment of the accuracy of "higher order theories"*

The kinematic relations (2)–(3) include trigonometric functions which account for finite rotations. If we substitute accurate values of trigonometric functions sin ϑ and cos ϑ with their approximations given by a truncated Taylor's series, 1st-order, 2nd-order, ...

Table 4. A buckling problem: a beam subjected at its end to a force P. One element, one load step

		(1)	(2)	(3)
$AE = 10^{21}$ $IE = 10$ $L = 1$				
e	n_G	ε_P (‰)	ε_P (‰)	ε_P (‰)
F3	3	0.27	89.60	—
	4	0.14	36.05	63.87
	10	0.14	36.05	63.87
F5	5	0.00	0.43	7.10
	6	0.00	0.18	0.59
	10	0.00	0.18	0.59
P_B *EXACT**		*24.674011*	*201.9073*	*394.784176*

e element type, F3 or F5.
n_G order of Gaussian integration.
ε_P relative error in P_B in comparison to exact solution, P_E; $\varepsilon_P = 1000(P_B - P_E)/P_E$, in ‰.
* Pflüger (1964).

theories are obtained. Thus an assessment of inaccuracy of the so-called "higher order theories" can be made. By 1st-order theory we define a theory which uses the approximation

$$\sin\vartheta \approx \vartheta \quad \cos\vartheta \approx 1. \tag{52}$$

This corresponds to the classical small displacement beam element. Second-order theory uses the relations

$$\sin\vartheta \approx \vartheta \quad \cos\vartheta \approx 1-\tfrac{1}{2}\vartheta^2. \tag{53}$$

Similarly, 3rd-order theory is introduced by the approximations

$$\sin\vartheta \approx \vartheta-\tfrac{1}{6}\vartheta^3 \quad \cos\vartheta \approx 1-\tfrac{1}{2}\vartheta^2+\tfrac{1}{24}\vartheta^4. \tag{54}$$

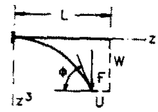
In Table 5 we compare the accuracy and convergence of a particular theory for the reduced loads $\bar{F} = 1$, 5 and 10 for element F5. First-order theory is clearly not adequate for describing large deformations. Second-order theory generally needs more iterations and displays the highest error up to 23% (compared to exact value) in the end deflection, W. The corresponding error of 3rd-order theory is only 2.6%; less than 0.001% error is found if accurate values of trigonometric functions are employed.

6. CONCLUSIONS

(1) We present a variational principle for finite planar deformations and rotations of a slender straight elastic beam, based on the principle of a stationary value of generalized potential energy of Hu–Washizu-type where the nonlinear kinematic equations are accounted for by Lagrangian multipliers. The Euler–Lagrange equations of the principle are exact kinematic and equilibrium equations and two equations for Lagrangian multipliers. By integrating the equations for Lagrangian multipliers and expressing the extensional strain of the centroid axis in terms of the axis rotation and Lagrangian multipliers, the functional of the variational principle is obtained which is a function of only one function of longitudinal coordinate, namely, the rotation of the centroid axis, and of eight discrete values, the boundary displacements/rotations and values of Lagrangian multipliers at $x = 0$. Since the first derivative of the rotation is the highest derivative in the functional, the C° continuity needs to be given only to the rotation function.

(2) Based on this variational principle and in conjunction with the iterative method of Newton, a family of new finite elements is proposed. Polynomial approximations of various

Table 5. A cantilever subjected at its free end to a force F. Assessment of "higher order theories". One element F5, one load step



$AE = 10^{21}$
$IE = 10$
$L = 1$
$\bar{F} = FL^2/EI = 0.1F$

$\bar{F}$	n_G	n_T	n_I	ε_u (‰)	ε_w (‰)	ε_ϕ (‰)
1.0	10	1	1	**−1000.00**	**104.77**	**83.78**
		2	5	**11.70**	**23.83**	**−1.08**
		3	4	**0.00**	**−0.20**	**0.02**
		A	4	**0.00**	**0.00**	**0.00**
	EXACT			*−0.05643*	*0.30172*	*0.46135*
5.0	10	1	1	**−1000.00**	**1334.96**	**1056.99**
		2	6	**29.54**	**160.22**	**−38.10**
		3	5	**−1.37**	**−11.63**	**2.02**
		A	5	**0.00**	**0.00**	**0.00**
	EXACT			*−0.38763*	*0.71379*	*1.21537*
10.0	10	1	1	**−1000.00**	**3112.13**	**2495.79**
		2	6	**21.24**	**233.37**	**−67.53**
		3	5	**−1.48**	**−26.46**	**5.85**
		A	5	**−0.02**	**−0.01**	**−0.01**
	EXACT ■			*−0.55500*	*0.81061*	*1.43029*
*EXACT SOLUTION**				*U/L*	*W/L*	ϕ

n_G order of Gaussian integration.
n_T order of theory; "A" means accurate trigonometric functions.
n_I number of Newton's iterations.
ε_Q relative error in quantity Q in comparison to exact solution, Q_E, $\varepsilon_Q = 1000(Q - Q_E)/Q_E$, in ‰.
* Mattiasson (1981).

orders for the rotation have been chosen. No approximation is necessary regarding the variation of displacements in the interior of an element. The example problems show that the accuracy of solution employing *only one element* to describe the deformations of a cantilever subjected to a lateral point load and *only one load step* for even extremely high loads, for the element with a polynomial of 5th-order for the rotation (named "element F5"), is really very high. The relative error in displacements and rotations compared with the exact solutions was less than 0.02‰ in all calculated examples. Element F5 behaves excellently in buckling, too; the error in buckling load for one element was found to be less than 1‰ for any combination of boundary conditions. The family of finite elements presented is *insensitive to the number of load steps* and the convergence proves to be quadratic. Gaussian integration of various orders has been used, but M-point integration for an element with $(M-1)$th order of polynomial for the rotation proves to be sufficient.

(3) The results indicate that the accuracy of the elements is not significantly influenced by the length of the elements, number of load steps and order of Gaussian integration (if it is greater than the minimal value, M). Due to the physical nature of Lagrangian multipliers, the influence of conservative distributed loads can be precisely accounted for. Therefore, a beam subjected to a variety of loads and extremely deformed, may be modeled with only one element, but still with a very high precision.

(4) For the sake of clarity, the present paper limits itself to the consideration of planar straight elastic beams with constant cross-section excluding shear deformations. A generalization to beams having varying cross-sections is a simple matter. The extension of the present variational principle to curved beams and inclusion of shear deformations are also straightforward. The author has achieved this employing the kinematic relations given by Reissner (1972) and the derivation will be presented elsewhere. Adaptation of the present variational principle to materials exhibiting nonlinear and creep effects is also feasible.

Acknowledgements—This work was supported by the Research Council of Slovenia, Yugoslavia, under Contract C2-3535-792. The author is grateful to Professor J. Banovec for useful discussions.

REFERENCES

Banovec, J. (1981). An efficient finite element method for elastic–plastic analysis of plane frames. In *Nonlinear Finite Element Analysis in Structural Mechanics* (Edited by W. Wunderlich, E. Stein and K.-J. Bathe), pp. 385–402. Springer, New York.

Epstein, M. and Murray, D. W. (1976). Large deformation in-plane analysis of elastic beams. *Comput. Struct.* **6**, 1–9.

Karamanlidis, D. (1988). Finite element analysis of two-dimensional shear flexible frame structures: nonlinear analysis. *Comput. Meth Appl. Mech. Engng* **67**, 55–68.

Mattiasson, K. (1981). Numerical results from large deflection beam and frame problems analysed by means of elliptic integrals. *Int. J. Numer. Meth. Engng* **16**, 145–153.

Noor, A. K. and Peters, J. M. (1981). Mixed models and reduced/selective integration displacements models for nonlinear analysis of curved beams. *Int. J. Numer. Meth. Engng* **17**, 616–631.

Pflüger, A. (1964). *Stabilitätsprobleme der Elastostatik*. Springer, Berlin, Heidelberg.

Reissner, E. (1972). On one-dimensional finite-strain beam theory: the plane problem. *J. Appl. Math. Phys. (ZAMP)* **23**, 795–804.

Saje, M. and Srpčič, S. (1985). Large deformations of in-plane beam. *Int. J. Solids Structures* **21**, 1181–1195.

Stein, E. (1982). Incremental methods in finite elasticity, especially for rods. In *Proc. IUTAM Symp. on Finite Elasticity* (Edited by D. E. Carlson and R. T. Shield), pp. 379–400. Martinus Nijhoff, The Hague, Boston, London.

Washizu, K. (1981). *Variational Methods in Elasticity and Plasticity*, 3rd edn. Pergamon Press, Oxford.

APPENDIX: COEFFICIENTS OF THE SYSTEM OF EQUATIONS (32)–(34)

$$\frac{\partial L_1}{\partial R_1^0} = -\frac{1}{AE}\int_0^L \cos^2\varphi \,\mathrm{d}x \tag{A1}$$

$$\frac{\partial L_1}{\partial R_2^0} = \frac{1}{AE}\int_0^L \sin\varphi\cos\varphi \,\mathrm{d}x \tag{A2}$$

$$\frac{\partial L_1}{\partial \varphi_m} = \int_0^L \left[(1+\varepsilon)\sin\varphi + \frac{1}{AE}Q\cos\varphi\right] I_m \,\mathrm{d}x \tag{A3}$$

$$\frac{\partial L_1}{\partial U_1} = -1 \tag{A4}$$

$$\frac{\partial L_1}{\partial U_4} = 1 \tag{A5}$$

$$\frac{\partial L_2}{\partial R_1^0} = \frac{\partial L_1}{\partial R_2^0} \tag{A6}$$

$$\frac{\partial L_2}{\partial R_2^0} = -\frac{1}{AE}\int_0^L \sin^2\varphi \,\mathrm{d}x \tag{A7}$$

$$\frac{\partial L_2}{\partial \varphi_m} = \int_0^L \left[(1+\varepsilon)\cos\varphi - \frac{1}{AE}Q\sin\varphi\right] I_m \,\mathrm{d}x \tag{A8}$$

$$\frac{\partial L_2}{\partial U_2} = -1 \tag{A9}$$

$$\frac{\partial L_2}{\partial U_5} = 1 \tag{A10}$$

$$\frac{\partial S_i^*}{\partial R_1^0} = \frac{\partial L_1}{\partial \varphi_i} \tag{A11}$$

$$\frac{\partial S_i^*}{\partial R_2^0} = \frac{\partial L_2}{\partial \varphi_i} \tag{A12}$$

$$\frac{\partial S_i^*}{\partial \varphi_m} = \int_0^L \left\{ EII_i' I_m' + \left[(1+\varepsilon)N - \frac{1}{AE} Q^2 \right] I_i I_m \right\} \mathrm{d}x. \tag{A13}$$

N and Q, the axial and shear forces in a cross-section, are given in eqns (49)–(50).